EXAM SYMMETRY IN PHYSICS

- Write your name and student number on the first page.
- Illegible writing will be graded as incorrect.
- Good luck!

Problem 1.

a. Consider the following Cayley table:

$G?$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	d	f	a	e	b
d	d	f	c	b	a	e
f	f	c	d	e	b	a

With this product, is the set $\{e, a, b, c, d, f\}$ a group? If so, is there a subgroup? If not, is there a subset which does form a group?

b. Let G be the group generated by the two matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

under matrix multiplication. Show that G is a nonabelian group of order 8. Is G isomorphic with D_4, the dihedral group, or with Q, the quaternion group?

c. Consider the permutation group S_4. Divide the group in conjugation classes and write down the different elements in cycle notation. How many (nonequivalent) irreps does S_4 have? Specify their dimensions.

d. Discuss the irreducible tensors of SU(3). Decompose the product representation $\mathbf{3}\otimes\mathbf{3}$ into irreps using the tensor method. Explain why no multiplet of hadrons corresponding to the SU(3)-flavor irrep $\mathbf{6}$ exists.

e. Discuss the structure of the Lie algebra of the Lorentz group. Give the generators. What are the Casimir operators? Discuss the irreps.

Problem 2.

The following character table of the group D_6 was *incorrectly* copied from a group theory book. There are two misprints in the labeling of the columns. Moreover, something else went wrong in the process of copying.

D_6	$\mathcal{E}$	$2C_6$	$2C_3$	C_2	$2C_2'$	$2C_2''$
A_1	1	1	1	1	1	1
A_2	1	1	1	1	−1	−1
B_1	1	−1	1	−1	1	−1
B_2	1	−1	1	−1	−1	1
E_1	2	1	−1	−2	0	0

The dihedral group D_6 is the symmetry group of a regular hexagon in the plane, containing one six-fold axis and six two-fold axes perpendicular to this principal axis. It is generated by the elements g and h, such that $g^6 = h^2 = e$ and $(gh)^2 = e$.

a. Express all the elements of D_6 in terms of g and h and give the conjugacy classes.

b. Correct the errors in the only possible way; explain your reasoning.

An atom with a valence electron is put in a hexagonal crystal environment with the symmetry of the dihedral group D_6 (ignore spin). For SO(3) the character of the conjugacy class labeled by the angle θ is

$$\chi^{(\ell)}(\theta) = \frac{\sin(2\ell+1)\theta/2}{\sin\theta/2} .$$

c. Calculate how the $\ell = 1$ and $\ell = 2$ states of a (central, spin-independent) potential are split by the crystal potential. Give the new degeneracies.

d. The hexagonal symmetry is further broken, by a trigonal distortion, to the subgroup D_3. Does this produce any further splitting?

e. Finally, the introduction of a magnetic field along the original six-fold axis reduces the symmetry to C_3. Without further calculation, state how the previous degeneracy is lifted.

Problem 3.

The proper Euclidean group E(3) in three space dimensions contains translations (beloning to the translation group T(3) in 3D) and rotations (belonging to the rotation group SO(3) in 3D), and specifically consists of the transformations:

$$\boldsymbol{r}' = R\boldsymbol{r} + \boldsymbol{a} \ , \qquad \text{with} \quad R \in \mathrm{SO}(3) \ , \quad \boldsymbol{a} \in \mathrm{T}(3) \ .$$

The elements of E(3) are written as $g = (\boldsymbol{a}, R)$.

a. What is the invariance connected to such transformations? Give the multiplication law, the unit element, and the inverse.

b. Which elements are conjugated? Give the conjugation classes.

c. Show that the four-dimensional matrix representation

$$g = \begin{pmatrix} R & \boldsymbol{a} \\ 0 & 1 \end{pmatrix}$$

is isomorphic to E(3).

d. For infinitessimal rotations and translations we can write

$$g = \exp(-iL) \simeq 1 - iL \ .$$

Show that L has the form

$$L = i \begin{pmatrix} B & \boldsymbol{a} \\ 0 & 0 \end{pmatrix} \ .$$

What conditions does the matrix B have to satisfy?

e. The matrices L are the elements of the Lie algebra of E(3). Choose a basis J_k, P_k $(k = 1, 2, 3)$ in this Lie algebra. Give the matrix form of J_k and P_k. Give the commutation relations.